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STRUCTURALISM AND THE NOTION OF DEPENDENCE

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The notion of dependence figures prominently in recent discussions of non-eliminative mathematical structuralism. Structuralists often argue that mathematical objects from one and the same structure depend on one another and on the structure to which they belong. Their opponents often argue that there cannot be any such dependence. I first show that the structuralists' claims about dependence are more important to their view than is generally recognized. Then I defend a compromise view concerning the dependence relations between mathematical objects, according to which the structuralists are right about some mathematical objects but wrong about others. I end with some remarks about the crucial notion of dependence.

The notion of dependence figures prominently in many recent discussions of non-eliminative mathematical structuralism. Supporters of this brand of structuralism often argue that mathematical objects from one and the same structure depend on one another and on the structure to which they belong. Their opponents often disagree, arguing instead that there cannot be any such dependence.

This paper has two goals. The first is to show that the structuralists' claims about dependence are more significant for their view than is generally recognized. I argue that these dependence claims play an essential role in the most interesting and plausible characterization of this brand of structuralism. The second goal is to defend a compromise view concerning the dependence relations that obtain between mathematical objects. Two extreme views have tended to dominate the debate, namely, the view that *all* mathematical objects depend on the structures to which they belong, and the view that *none* does. I present counter-examples to each of these extreme views. I defend instead a compromise view, according to which the structuralists are right about many kinds of mathematical objects (roughly, the algebraic ones), whereas the anti-structuralists are right about others (in particular, the sets). I end with some remarks about how to understand the crucial notion of dependence, which despite being at the heart of the debate is rarely examined in any detail.

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I. WHAT IS NON-ELIMINATIVE MATHEMATICAL STRUCTURALISM?

I begin by making clear which brand of structuralism will occupy me. The first thing to note is that this brand is a version of *mathematical* structuralism. Very roughly, mathematical structuralism is the view that pure mathematics is the investigation of abstract structures, and that all that matters to mathematics is purely structural properties of objects. Mathematical structuralism is thus a local form of structuralism, restricted to the field of mathematics. This contrasts with various global forms of structuralism, which attempt to say something about objects or our cognitive relation to them quite generally.[1] It also contrasts with other local forms of structuralism, for instance, structuralism about the physical world.[2] Mathematical structuralism is distinct from and independent of these other forms of global and local structuralism.

The next important distinction is between *eliminative* and *non-eliminative* versions of mathematical structuralism. It is perhaps easiest to begin by describing the non-eliminative version. This takes structuralism to be a fundamental insight about the nature of mathematical objects, namely, that these objects are really just positions in abstract mathematical structures. The natural number 2, for instance, is just the second (or on some approaches, the third) position in the abstract structure instantiated by all systems of objects satisfying the second-order Dedekind–Peano axioms.

The eliminative versions of mathematical structuralism are unified mostly by their opposition to the non-eliminative version just outlined. The eliminative versions deny that there are abstract mathematical structures and that the nature of mathematical objects is exhausted by their being positions in such structures. Three versions of eliminative structuralism can be distinguished. *Deductivist structuralism* avoids both ontological commitment to mathematical objects and all use of modal vocabulary. It interprets mathematics as the formulation of various (mostly categorical) theories to describe various kinds of concrete structures, and as the study of what holds in all models of each of these theories.[3] *Modal structuralism* lifts the

[1] See W.V. Quine, 'Structure and Nature', *Journal of Philosophy*, 89 (1992), pp. 5–9, for an example of a semantic form of global structuralism.

[2] See J. Ladyman, 'What is Structural Realism?', *Studies in the History and Philosophy of Science*, 29 (1998), pp. 409–24, for an example of structuralism about the physical world.

[3] Canonical examples are B. Russell, *Principles of Mathematics* (New York: Norton, 1903); H. Putnam, 'The Thesis that Mathematics is Logic', repr. in his *Mathematics, Matter and Method* (Cambridge UP, 1975), pp. 12–42.

deductivist ban on modal notions. It interprets mathematics as asserting that it is possible for various theories to have concrete models, and as studying what necessarily holds in all such models.[4] Finally, *set-theoretic structuralism* rejects the deductivist nominalism in favour of a background theory of sets. It then takes mathematics to be the study of various structures realized among the sets. This is often what mathematicians have in mind when they talk about structuralism.

In what follows, my sole concern will be with non-eliminative mathematical structuralism. I shall therefore reserve the word 'structuralism' (and 'structuralist', etc.) for this view.

A fundamental question that confronts structuralism is how this view differs from more traditional Platonist views of mathematics. After all, both types of view are ontologically committed to abstract mathematical objects, and both agree that these objects enter into larger structures. An initial response from structuralists is that on their view, mathematical objects are *nothing more than* positions in abstract structures, whereas on traditional Platonist views, mathematical objects have a richer nature than this. But although suggestive, this initial response needs to be spelt out. It is of limited use to be told that mathematical objects are identical with positions in abstract structures before an account has been given of the nature of these positions, and in particular of how they differ from mathematical objects as conceived by the traditional Platonist.

Fortunately, structuralists make a variety of more substantive claims about how their view differs from more traditional Platonist views. Two main claims stand out. First, what I shall call 'the incompleteness claim' says that mathematical objects are incomplete in the sense that they have no 'internal nature' and no non-structural properties. The following passage from a seminal early article on structuralism gives a typical expression of this claim:

> In mathematics, I claim, we do not have objects with an 'internal' composition arranged in structures, we have only structures. The objects of mathematics ... are structureless points or positions in structures. As positions in structures, they have no identity or features outside a structure.[5]

Secondly, what I shall call 'the dependence claim' says that mathematical

[4] An early example is Putnam, 'Mathematics without Foundations', *Journal of Philosophy*, 64 (1967), pp. 5–22. The most developed version is G. Hellman, *Mathematics without Numbers* (Oxford: Clarendon Press, 1989).

[5] M. Resnik, 'Mathematics as a Science of Patterns: Ontology and Reference', *Noûs*, 15 (1981), pp. 529–50, at p. 530, a passage quoted approvingly in a large number of defences of structuralism, e.g., C. Parsons, 'The Structuralist View of Mathematical Objects', *Synthese*, 84 (1990), pp. 303–46, at p. 303; S. Shapiro, *Philosophy of Mathematics: Structure and Ontology* (Oxford UP, 1997), p. 75.

objects from one structure are dependent on one another and/or on the structure to which they belong. This claim is meant to conflict with a more traditional Platonist view, which ascribes a greater degree of independence to mathematical objects. The following passage provides a good example:

> The number 2 is no more and no less than the second position in the natural number structure; and 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as positions in this structure, neither number is independent of the other.[6]

II. THE INCOMPLETENESS CLAIM

As I mentioned in the introduction, my main concern in this paper will be with the dependence claim. But before I embark on any attempt to analyse and assess this, it makes sense to pause and discuss whether the claim really matters. What if the claim is just some throw-away remark, intended to carry little or no weight?

There are at least two reasons why the dependence claim deserves to be taken seriously. The most obvious reason is simply that it is an interesting metaphysical claim, to which almost all structuralists (explicitly or implicitly) commit themselves. This will be discussed in the next section. A less obvious reason, which will be my concern in this section, is that the incompleteness claim is deeply problematic. Although I shall not attempt any conclusive assessment of these problems, I shall argue that they are serious enough to make it very unwise for structuralists not to take the dependence claim seriously.

The literature on structuralism is replete with statements of various forms of the incompleteness claim. Some of these statements have to do with there being no 'fact of the matter' about cross-category identity statements, such as whether the natural number 2 is identical with a certain set, or with Julius Caesar.

> Mathematical objects are incomplete in the sense that we have no answers within or without mathematics to questions of whether the objects one mathematical theory discusses are identical to those another treats.[7]

> ... it makes no sense to pursue the identity between a place in the natural-number structure and some other object, expecting there to be a fact of the matter. Identity between natural numbers is determinate; identity between numbers and other sorts of objects is not.[8]

[6] Shapiro, *Thinking about Mathematics* (Oxford UP, 2000), p. 258.
[7] Resnik, *Mathematics as a Science of Patterns* (Oxford UP, 1997), p. 90; see also p. 210.
[8] Shapiro, *Philosophy of Mathematics: Structure and Ontology*, p. 79; see also pp. 80–1, 258ff.

For this reason mathematical objects are often said to be 'incomplete', somewhat like fictional objects.[9] Other statements claim that the objects of mathematics are structureless points with no internal nature:

> Mathematics is concerned with structures involving mathematical objects and not with the 'internal' nature of the objects themselves.[10]
>
> [There] is no more to the individual numbers 'in themselves' than the relations they bear to each other.[11]
>
> The idea behind the structuralist view of mathematical objects is that such objects have no more of a 'nature' than is given by the basic relations of a structure to which they belong.[12]

Examination of these quotations shows that the incompleteness claim consists of two *prima facie* different strains. According to the first, mathematical objects possess no *non-structural properties*. According to the second strain, they have no *internal composition*, and more generally, no *intrinsic properties*. I shall refer to these two strains as 'NS-incompleteness' and 'I-incompleteness' respectively. Depending on what the relation is between the non-structural and the intrinsic properties of mathematical objects, these two strains may or may not be extensionally equivalent. But since the strains are certainly intensionally different, I shall discuss them separately.

When NS-incompleteness denies that mathematical objects have any non-structural properties, what exactly is meant by a 'structural property'? A useful answer to this question can be extracted from the work of Shapiro. He uses (*Philosophy of Mathematics*, p. 73) the notion of a *system* to refer to 'a collection of objects with certain relations'. Following Shapiro, I shall use a separate kind of upper-case variable to range over 'collections' of objects. For present purposes I do not need to take a stand on how these variables (and the quantifiers binding them) are to be interpreted. (The options include pluralities, sets, classes, and properties.) A system in Shapiro's sense is then an ordered $(n+1)$-tuple consisting of a domain D and relations on this domain $R_1, \ldots, R_n$. (There is no need to have separate variables for designated objects, as these can if necessary be represented by means of the relations, by using, for each object, a one-place relation uniquely true of it.)

Next, Shapiro (p. 74) introduces the notion of a structure as 'the abstract form of a system, highlighting the interrelationships among the objects, and

[9] See, e.g., Parsons, 'Mathematical Intuition', *Proceedings of the Aristotelian Society*, 80 (1979/80), pp. 145–68, at §II, and 'The Structuralist View of Mathematical Objects', pp. 334–5; Resnik, *Mathematics as a Science of Patterns*, pp. 90, 211.

[10] Resnik, 'Mathematics as a Science of Patterns: Ontology and Reference', p. 529.

[11] Shapiro, *Philosophy of Mathematics: Structure and Ontology*, p. 73.

[12] Parsons, 'Structuralism and Metaphysics', *The Philosophical Quarterly*, 54 (2004), pp. 56–77, at p. 57.

ignoring any features of them that do not affect how they relate to other objects in the system'. I shall refer to this process as *Dedekind abstraction*, in honour of one of its earliest and most important defenders.[13] Although Shapiro's characterization of this relies heavily on epistemological language, it is clear that the process itself is supposed to be a metaphysical one (a thoroughly metaphysical analysis of it will be proposed in §VI below).

A *structural property* can now be characterized as a property that can be arrived at through this process of abstraction, or, equivalently, a property that is shared by every system that instantiates the structure in question. The NS-incompleteness claim then says that the only properties that mathematical objects possess are the ones that are structural in this sense.

However, the NS-incompleteness claim, stated thus boldly, is exposed to a variety of counter-examples.[14] For instance, the number 8 has the property of being my favourite number. It also has the property of being the number of books on one of my shelves. It also has non-structural properties such as being abstract and being a natural number. In fact, the property of being abstract seems to be a very important property of natural numbers. A student of arithmetic who has not realized that numbers are abstract is missing something fundamental. Finally, as John Burgess observes (p. 286), it is doubtful that the NS-incompleteness claim can even be coherently stated. For the property of having only non-structural properties is not itself a structural property, since it is not shared by all systems that instantiate the structure in question.

Because of these problems, at least one important structuralist, Stewart Shapiro, has recently backed off from most aspects of the incompleteness claim. Shapiro now argues that mathematical objects from distinct structures are always (determinately and objectively) distinct; and he comes close to accepting that mathematical objects possess non-structural properties such as being abstract.[15]

Can the idea behind the NS-incompleteness claim be salvaged by being given a more modest formulation?[16] Instead of claiming that absolutely all properties of mathematical objects are purely structural, perhaps

[13] This label is used in Parsons, 'The Structuralist View of Mathematical Objects'; he attributes it to William Tait.

[14] See J.P. Burgess, review of Stewart Shapiro, *Philosophy of Mathematics: Structure and Ontology*, *Notre Dame Journal of Formal Logic*, 40 (1999), pp. 283–91, at p. 286; Ø. Linnebo, Critical Notice of Stewart Shapiro, *Philosophy of Mathematics: Structure and Ontology*, *Philosophia Mathematica*, 11 (2003), pp. 92–104, at pp. 97–9; F. MacBride, 'Structuralism Reconsidered', in S. Shapiro (ed.), *Oxford Handbook of Philosophy of Mathematics and Logic* (Oxford: Clarendon Press, 2005), pp. 563–89, at pp. 583–4.

[15] See Shapiro, 'Structure and Identity', in F. MacBride (ed.), *Identity and Modality* (Oxford: Clarendon Press, 2006), pp. 109–45, at pp. 121–31.

[16] See Shapiro, 'Structure and Identity', p. 115.

structuralists can restrict the claim to some important kind of properties. But what could this important kind of properties be? Given that abstractness is presumably a necessary property of mathematical objects, it would not suffice to restrict the claim to necessary properties. The only restriction that might work, it seems, would be to those properties of a mathematical object that *matter for its identity*. But this would transform the NS-incompleteness claim to something close to the dependence claim. For instance, the claim might then be that the identity of a mathematical object is grounded in its structural properties, and that the object in this sense depends upon the structure to which it belongs.

Turning now to the I-incompleteness claim, I first remark that any 'internal composition' of an object would have to be reflected in its intrinsic properties. It is therefore sufficient to consider the claim that mathematical objects have no intrinsic properties. But this claim too is problematic. Here the problem is to make sense of what it would mean for a mathematical object to have intrinsic properties. An intrinsic property is supposed to be a property that an object has solely in virtue of what it is like and not in virtue of the relations it bears to the rest of reality. In the literature there are two main ways of spelling out the notion of an intrinsic property.[17] On the first analysis, a property is intrinsic to an object if and only if it is shared by every *duplicate* of the object. But this analysis is of little use in connection with mathematics because it is hard to make sense of the notion of a duplicate of a mathematical object. Mathematical objects simply are not a kind of things we ordinarily think of as having duplicates. (If a mathematical object has no duplicates except itself, then *all* of its properties will count as intrinsic, in diametrical opposition to the I-incompleteness claim.) But I shall temporarily waive this worry and ask what would be required of a duplicate relation for it to validate the I-incompleteness claim.[18] The answer is that each mathematical object would have to have duplicates which are so diverse that they do not share even a single property. This would mean that the duplicates of, say, a natural number would have to include non-numbers, and even objects that are not abstract! I conclude that on this first approach to intrinsicness we would be forced to operate with a duplicate relation which both is poorly understood and would clash with the few intuitions we may have about it.

On the second analysis, a property is intrinsic to an object if and only if the object would have this property even if the rest of the universe were

[17] See B. Weatherson, 'Intrinsic *vs* Extrinsic Properties', in E.N. Zalta (ed.), *Stanford Encyclopedia of Philosophy*, http://plato.stanford.edu/archives/fall2006/entries/intrinsic-extrinsic.

[18] Thanks to Ross Cameron and Robbie Williams for questions that forced me to clarify this.

removed or disregarded. This is somewhat more promising. For on this analysis, the I-incompleteness claim can be understood as the claim that no mathematical object can possess properties completely on its own: rather, a property can meaningfully be ascribed to a mathematical object only when this object is regarded as part of the structure to which it belongs. However, thus understood, the I-incompleteness claim comes to little more than the dependence claim. For to claim that an object cannot be considered in isolation from its structure, and that it cannot possess any properties except as a component of this structure, is to make a claim about how the object *depends upon* this structure.

To sum up, I have shown that the first strain of the incompleteness claim faces serious difficulties, which have led at least one prominent structuralist to back away from it. I have also argued that any attempt to salvage the first strain or make sense of the second results in claims which are very similar to the dependence claim. These findings give structuralists and non-structuralists alike a good reason for taking the dependence claim seriously.

III. THE DEPENDENCE CLAIM

Almost all structuralists commit themselves to the dependence claim, either explicitly or implicitly. I have already pointed out that this claim plays an important role in the characterization of structuralism offered in Shapiro's *Thinking about Mathematics*. But the claim figures prominently in his more scholarly *Philosophy of Mathematics* as well. Here is one example (p. 78):

> The structure is prior to the mathematical objects it contains, just as any organization is prior to the offices that constitute it.

More generally, Shapiro's *Philosophy of Mathematics* contrasts structuralism with traditional Platonism, which (he says) assimilates mathematical objects to ordinary physical objects. One fundamental feature of ordinary physical objects is that they are supposed to be ontologically independent of one another. For instance, my computer is supposed to be capable of existing even if my desk did not, and *vice versa*. According to Shapiro (p. 72), the Platonist claims that the same holds of mathematical objects, for instance, that 'each natural number is independent of every other natural number'. Structuralism can then be distinguished from traditional Platonism in that it denies that mathematical objects from the same structure are ontologically independent of one another.

But as Shapiro notes, it is difficult to spell out the relevant notion of independence in a satisfactory way. He takes some initial steps towards an explication, relying on the notion of *essence*. In particular, he suggests that the Platonist view 'may be that one can state the essence of each number without referring to the other numbers'. He then comments (*ibid.*) that

> If this notion of independence can be made out, we structuralists would reject it. The essence of a natural number is its relations to other natural numbers.

Similar claims that the natural numbers are essentially related to one another are found in other places, for instance (p. 5),

> ... the essence of a natural number is the relations it has with other natural numbers.

The essence of an object is normally understood as what gives the object its identity, what makes it the object that it is. Shapiro thus appears to be saying that the identity of a mathematical object depends on or is derived from the relations that it bears to other mathematical objects from the same structure. This means that a mathematical object has its identity only in virtue of occupying a certain position in the structures to which it belongs. In this sense the object presupposes or depends on the structure to which it belongs. By contrast, ordinary concrete objects do not in this way presuppose or ontologically depend on the various structures into which they enter. For instance, although an apple occupies positions in various geometrical, biological and economic structures, it is generally assumed that the apple has its identity independently of all these facts, and is such that it could occupy other positions in such structures, or perhaps not even enter into any structure at all, while still remaining the object that it is.

Michael Resnik too commits himself to the dependence claim, although somewhat less explicitly than Shapiro does. For instance, the following passage contains the same idea as in Shapiro, namely, that mathematical objects owe their identities to their structural relationships to other mathematical objects:

> Mathematical objects ... have their identities determined by their relationships to other positions in the structure to which they belong.[19]

I shall now try to be more precise about what the dependence claim says. One aspect of the dependence claim is that in the domain D of some mathematical structure, *objects depend on objects*:

ODO. Each object in D depends on every other object in D.

For instance, each natural number is said to depend on all the other natural numbers in a way in which it does not depend on, say, sets. It follows that it

[19] Resnik, 'Mathematics as a Science of Patterns: Epistemology', *Noûs*, 16 (1982), pp. 95–105, at p. 95.

is impossible for one natural number to exist without all the others existing as well. Another aspect of the dependence claim is that in mathematical structures *objects depend on structures*:

ODS. Each mathematical object depends on the structure to which it belongs.

(ODS) has much the same effect as (ODO). In particular, (ODS) ensures that it is impossible for one object from a particular mathematical structure to exist without all the other objects from this structure existing as well. For if one object from some structure exists, then by (ODS) so too does its structure. But since a structure involves all of its positions, a structure cannot exist without all of its positions existing as well. Putting these claims together, it follows that if one object from a mathematical structure exists, then so do all the others.

If defensible, (ODO) would be a very substantial metaphysical discovery. It would mean that mathematical objects are subject to an *upwards dependence*, in that they depend on the structures to which they belong. If correct, this may point to a fundamental difference between the mathematical realm and the physical, where an opposite form of *downwards dependence* appears to dominate. For most structures composed of physical objects appear to depend on their constituents, rather than the other way round (though there may well be exceptions: for instance, a quantum particle in an entangled state is arguably ontologically dependent on the entire entangled system).

Both of the claims (ODO) and (ODS) are endorsed by Stewart Shapiro, who uses them as a cornerstone of his characterization of structuralism. I have given one clear example of this in the following passage from *Thinking about Mathematics*, p. 258, quoted above in §I:

> The number 2 is no more and no less than the second position in the natural number structure; and 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as positions in this structure, neither number is independent of the other.

IV. RECENT OBJECTIONS TO THE DEPENDENCE CLAIM

Several philosophers have recently objected to the dependence claim. Before I discuss these objections it will be instructive to consider a related but somewhat simpler objection, which as far as I know has not been developed in print. This objection is based on the claim that there cannot be circular relations of dependence. The objector can attempt to support this claim by observing that when an object b depends on another object a, then *a* must be 'prior' to *b*. But then *a* cannot in turn depend on *b*, since two objects cannot be 'prior' to each other in one and the same sense. The objector thus arrives at the claim that there cannot be any cyclical relations of dependence. The objector may also attempt to argue for the stronger claim that the notion of dependence must be well founded:

WF. The dependence relation must be well founded.[20]

Armed with either the non-circularity requirement or the stronger well foundedness requirement, the objector can reject the dependence claim (ODO) as unacceptable. For this claim says that mathematical objects stand in circular relations of dependence.

However, even if this objection could be made out – and that is a very big 'if' – it is doubtful whether it would do much damage to structuralism. For even if forced to give up (ODO), structuralists would still have the closely related dependence claim (ODS) to fall back on. This is because (ODS) postulates only a one-way dependence of objects upon the structure to which they belong. There is thus no obvious conflict between (ODS) and the requirement that the dependence relation must be non-circular or even well founded. Moreover, even on its own (ODS) gives structuralists almost everything that they want. For as I have said, (ODS) is just as effective as (ODO) in ensuring that if one object from a mathematical structure exists, then so do all other objects from this structure. Furthermore, because (ODS) is a claim about upwards dependence, it allows structuralists to draw a sharp dividing line between mathematical objects and ordinary concrete objects, which tend to be subject to a downwards dependence.

Geoffrey Hellman and Fraser MacBride have developed more sophisticated objections intended to undermine not just (ODO) but also (ODS). According to these objections, even the upwards dependence of (ODS) would introduce an impermissible circular dependence. Like the simple objection just discussed, Hellman's and MacBride's objections begin by assuming that the dependence relation must be well founded, or at least non-circular. They then continue by defending (two different versions of) a completely general metaphysical thesis of downwards dependence, roughly

[20] E.J. Lowe, in his 'Individuation', in M. Loux and D. Zimmerman (eds), *Oxford Handbook of Metaphysics* (Oxford UP, 2003), pp. 75–95, and 'Ontological Dependence', in Zalta (ed.), *Stanford Encyclopedia of Philosophy*, http://plato.stanford.edu/archives/sum2005/entries/dependence-ontological, defends the ban on cyclical relations of dependence. K. Fine, 'Ontological Dependence', *Proceedings of the Aristotelian Society*, 95 (1995), pp. 269–90, defends a qualified form of the principle (WF). This principle is also implicit in Hellman, 'Three Varieties of Mathematical Structuralism', *Philosophia Mathematica*, 9 (2001), pp. 184–211, and in MacBride, 'What Constitutes the Numerical Diversity of Mathematical Objects?', *Analysis*, 66 (2006), pp. 63–9, to be discussed below.

to the effect that any structure whatsoever depends on the objects that it involves. If (either version of) this general thesis is correct, then this will undermine (ODS) as well. But I shall now go on to argue that both versions of the thesis of universal downwards dependence are highly problematic.

Hellman's argument focuses on one particular example, namely, the abstract structure shared by all systems that satisfy the Peano–Dedekind axioms. According to structuralists, the positions in this abstract structure

> are entirely determined by the successor function ... and derivative from it in the sense of being identified merely as the terms of the ordering induced by [this successor function].[21]

Provided that the crucial words 'determined' and 'identified' are understood in a metaphysical rather than an epistemological sense, this is a fair characterization of the structuralist view. But Hellman has misgivings about this view, as brought out (*ibid.*) in the following question:

> But if the relata are not already given but depend for their very identity upon a given ordering, what content is there to talk of 'the ordering'?

After expressing doubts about whether structuralists are in a position to answer this question, Hellman concludes as follows:

> This, I submit, is a vicious circularity: in a nutshell, to understand the relata, we must be given the relation, but to understand the relation, we must already have access to the relata.[22]

This conclusion is confusing, given the unexplained shift from metaphysical to epistemological vocabulary. Since the structuralists clearly intend the dependence claim to be a metaphysical claim, the most charitable reading of the passage will presumably be to translate it back into purely and unambiguously metaphysical vocabulary. I thus propose to interpret Hellman's claims about what is required in order to understand an entity as claims about how the identity of this entity is *grounded*. When the above passage is translated in this way, it yields something like the following:

> This, I submit, is a vicious circularity: the structuralists claim that the identity of the relata is grounded in that of the relation; but any grounding of the identity of a relation presupposes that the relata have already and independently had their identities grounded.

If this interpretation is on the right track, then Hellman is relying on the following premise, according to which relations in a certain sense depend upon their relata:

[21] Hellman, 'Three Varieties of Mathematical Structuralism', pp. 193–4.

[22] Another similar passage is found in Hellman, 'Structuralism', in Shapiro (ed.), *Oxford Handbook of the Philosophy of Mathematics and Logic*, pp. 536–62, at p. 545.

RDO_1. The identity of any relation on a domain D presupposes that the individual objects from D have already and independently had their identities grounded.

This premise is potentially very damaging to structuralism. For a structure is a system of abstract objects that stand in purely structural relationships to one another. So if the relations involved in the structure depend on their relata as asserted by (RDO_1), then so will the structure. Given the requirement that the dependence relation must be well founded, this downwards dependence would be incompatible with the upwards dependence of (ODS).

However, no defence of the thesis (RDO_1) is provided. Perhaps the thesis is meant to be intuitively obvious. If so, structuralists can respond that this intuition is just wrong, and that mathematics provides counter-examples. Or perhaps the thesis is meant to be supported by examples. For instance, the identity of the relation of *being cousins* seems to presuppose that the relata, human beings, have already and independently had their identities grounded. For in order to say anything substantive about what relation this is, we will have to specify how two human beings, already individuated, have to be related in order for the relation to apply. However, this defence of (RDO_1) would plainly beg the question against the structuralists, whose point is precisely that mathematical objects differ from ordinary physical objects by being subject to an upwards rather than a downwards dependence. I conclude that no satisfactory defence of (RDO_1) has been offered.

MacBride's argument, which he develops without fully endorsing it,[23] differs somewhat from Hellman's. It is summarized in the following passage:

> In order for objects to be eligible to serve as the terms of a ... relation they must be independently constituted as numerically diverse. Speaking figuratively, they must be numerically diverse 'before' the relation can obtain; if they are not constituted independently of the obtaining of a ... relation then there are simply no items available for the relation in question to obtain between.[24]

The crucial premise employed here is a second form of universal downwards dependence:

RDO_2. The obtaining of any relation presupposes that the objects it relates have already had their identities grounded.

Much the same can be said about this premise as about (RDO_1): although it would clinch the anti-structuralist argument, no defence of the premise is

[23] Indeed, in his 'Structuralism Reconsidered', in Shapiro (ed.), *Oxford Handbook of Philosophy of Mathematics and Logic*, pp. 563–89, at p. 581, MacBride expresses serious reservations about this kind of argument.

[24] See MacBride, 'What Constitutes the Numerical Diversity of Mathematical Objects?', p. 67.

provided, and any appeal to intuitive obviousness or to examples would have little suasive power *vis à vis* structuralism.

V. A COUNTER-EXAMPLE TO THE DEPENDENCE CLAIM

I have remarked that two extreme views tend to dominate the debate: the structuralist claim that *all* mathematical objects depend on the structures to which they belong, and the anti-structuralist claim that *no* mathematical object is subject to any such upwards dependence. I shall now provide counter-examples to each of these extreme views. My own position will be a compromise according to which some mathematical objects depend on their structures, whereas others do not.

This sort of compromise view is not new. Kant distinguished between two sorts of totalities with different dependence properties.[25] A *totum syntheticum* is a totality 'synthesized' from pre-existing parts, which therefore depends on these parts. A *totum analyticum*, on the other hand, is a totality which is prior to its constituents and whose constituents therefore depend on the whole. Kant's principal examples of this sort of totality were space and time. For instance, he argued that the totality of space is prior to all of its different sub-spaces. A more recent example is found in the work of Charles Parsons. After articulating and defending a structuralist view of 'pure' mathematical objects, he argues that structuralism does not give a correct description of more basic '*quasi*-concrete' mathematical objects, such as geometrical figures and linguistic types, which have canonical representations in the concrete.[26] In order to refute the two extreme views, it would suffice to defend either of these two earlier compromise views. However, I shall be concerned with different examples, and I shall finish by demarcating the scope of structuralism in a new and different way.

I believe that sets provide examples of mathematical objects that are not subject to the upwards dependence of the theses (ODO) and (ODS). According to the prevailing iterative conception, sets are 'formed from' their elements. The relation between a set and its elements is thus asymmetric, because the elements must be 'available' before the set can be formed, whereas the set need not be, and indeed cannot be, 'available' before its elements are formed. A set thus appears to depend on its elements in a way in which the elements do not depend on the set.[27]

[25] See H. Allison, *Kant's Transcendental Idealism* (Yale UP, 1986), p. 43, citing Kant's *Reflexion* 393.

[26] See Parsons, 'The Structuralist View of Mathematical Objects', pp. 337–8.

[27] See G. Boolos, 'The Iterative Conception of Set', *Journal of Philosophy*, 68 (1971), pp. 215–32. The asymmetric dependence of a set on its elements is also defended in Fine, 'Ontological Dependence'; Lowe, 'Ontological Dependence'.

This asymmetry is perhaps most striking in the case of singletons. The identity of a singleton depends on that of its single element. For in order to specify what singleton we are talking about, we need to specify what its single element is. But the identity of the singleton does not depend on any other objects or on the hierarchy of sets. For an exhaustive account can be given of the identity of the singleton without mentioning any objects other than its single element.

This asymmetric dependence is in fact a very good thing, as there are all kinds of difficult questions about the higher reaches of the hierarchy of sets. How far does the hierarchy extend? Are the different stages rich enough for the continuum hypothesis to fail? It would be a pity if very simple sets, such as the empty set and its singleton, depended on the entire hierarchy of sets, and their identities could therefore not be completely known before these hard questions had been answered. But fortunately the situation is the reverse. In particular, we can give an exhaustive account of the identity of the empty set and its singleton without even mentioning infinite sets. (I suspect it is possible to defend an analogous view of the natural numbers, according to which one natural number depends on its predecessors but not *vice versa*.)

How can the structuralist respond to this argument? One response would be to argue that we should abandon the iterative conception of set and adopt some alternative structuralist conception according to which all sets are individuated simultaneously. Although I am inclined to concede that such an alternative structuralist conception of set is possible, I do not think this response has much force. For my present claim is simply that the ordinary iterative conception of set provides us with a counter-example to the dependence theses (ODO) and (ODS). This claim cannot be undermined by the development of any *alternative* conception of set.

A second and more promising response would be to challenge my claim that the ordinary iterative conception of set is non-structuralist. This strategy is pursued by Parsons in a response to the present problem, so far as I know the only one to be found in the literature.[28] He first argues that ZFC set theory rests on several distinct intuitions and cannot be reduced to any one of these. First, there are combinatorial intuitions, associated with the conception of sets as collections. These intuitions support an ontologically rich membership relation which sustains an asymmetric dependence of a set upon its elements. Next, there is the idea of limitation of size, which plays an important role in the justification of the axiom of replacement. Parsons then argues that since no one unified set of intuitions can be used to support all of

[28] See Parsons, 'Structuralism and the Concept of Set', in W. Sinnott-Armstrong (ed.), *Modality, Morality, and Belief* (Cambridge UP, 1994), pp. 74–92.

ZFC, a structuralist conception of sets is more plausible. However, even if Parsons is right that ZFC cannot be supported by one unified set of intuitions, this would not undermine my present argument. For all I need is that one strand of the iterative conception provides a non-structuralist conception of sets, not that this strand suffices to generate *all* the axioms of ZFC. It seems clear that the combinatorial intuitions suffice to ground at least the theory of hereditarily finite sets, which I can then use as my counter-example.

A third response would be to scrutinize the notion of dependence rather than the iterative conception of set. Perhaps a better understanding of the notion of dependence will show that sets do not constitute a counter-example after all? I agree that the intuitive notion of dependence which I have relied on needs to be analysed. I therefore propose an analysis in §VII below, and show there that on this analysis, sets do indeed constitute a counter-example to the dependence claims (ODO) and (ODS). Structuralists may obviously develop an alternative analysis of the notion of dependence. But until I am actually presented with one, all I can do is express my grave doubts that any such analysis can manage to overturn our strong conviction that sets depend asymmetrically on their elements.

VI. EXAMPLES WHERE THE DEPENDENCE CLAIM HOLDS

I shall now describe some examples of mathematical objects of which the dependence claim is true. These will also serve as counter-examples to the anti-structuralist downwards-dependence theses (RDO_1) and (RDO_2). I begin with a loose and intuitive treatment of a simple example, before turning to a more systematic exposition. In the unique group with two elements, the elements may be denoted by α and β, and an exhaustive characterization of the group may be given by means of the multiplication table in Figure 1. Provided we are willing to accept α and β as *bona fide* mathematical objects, this provides an example where the dependence claim holds. (ODO) holds because α and β are individuated simultaneously and in relation to each other. Each object is what it is only by entering into certain relationships with the objects that make up the group. (ODS) holds for much the same reason: since α and β are nothing but positions in this structure, their identity cannot be characterized without also characterizing the identity of the entire structure.

*	α	β
α	α	β
β	β	α

Figure 1

I shall now do things a bit more carefully and generally. (Readers who are less technically inclined may prefer to skip to the last two paragraphs of this section.) Let R and R' be n-place relations. I shall say that R and R' are *isomorphic* iff there is a one-to-one mapping f from the field of R onto the field of R' such that $Rx_1 \ldots x_n$ iff $R'f(x_1) \ldots f(x_n)$. I symbolize this as $R \cong R'$. I can now define *isomorphism types* of relations by the following abstraction principle:

1. $\bar{R} = \bar{R}' \leftrightarrow R \cong R'$.

But one has to be careful here, for without any restrictions, (1) leads straight to paradox.[29] Fortunately, a variety of restrictions are known which would shield the account from paradox. The simplest restriction is just to let the relation-variables range only over sets and to let their isomorphism types be non-sets. Although this is not entirely satisfactory in general, it suffices for my present goal of providing examples of one-way upwards dependence.[30]

I next claim that the isomorphism types of relations can be used to represent what I called in §III *Dedekind abstraction*, that is, the operation that maps a system to its abstract structure. As I have said, Shapiro characterizes the structure of a system based on a domain D and relations $R_1, \ldots, R_n$ as 'the abstract form' of this system. Let R be the product relation $R_1 \times \ldots \times R_n$ (where the product $S \times T$ of an m-place relation S and an n-place relation T is the $m+n$-place relation that holds of $x_1, \ldots, x_{m+n}$ if and only if S holds of $x_1, \ldots, x_n$ and T holds of $x_{m+1}, \ldots, x_{m+n}$). The single relation R can be regarded as a complete representation of the entire system with which I began, including all of the relations $R_1, \ldots, R_n$. This is made clear by the example of an algebraic ring. (A ring is a system consisting of a domain D on which are defined three-place relations Add and Mult such that the usual axioms for addition and multiplication are satisfied. Paradigm examples are the familiar number systems Z, Q, R and C.) But the system associated with a ring is completely represented by means of an 'addition table' and a 'multiplication table' defined on the relevant domain. The product relation R is just a way of combining these two tables into one.

Given this way of representing particular systems, their structure or abstract form can be represented simply as $\bar{R}$. Where the particular system R has offices that are filled by independently individuated occupants, $\bar{R}$ is left with nothing but the offices themselves: everything else has been abstracted away from. The construction of a 'product system' is needed in order to keep track of how the same offices figure in the isomorphism types of the

[29] It will allow us to derive Burali-Forti's paradox: see A.P. Hazen, review of Crispin Wright, *Frege's Conception of Numbers as Objects*, *Australasian Journal of Philosophy*, 63 (1985), pp. 250–4, at pp. 253–4.

[30] For a more systematic solution, see Ø. Linnebo, 'Bad Company Tamed', forthcoming in *Synthese*, 2008.

various relations R_i even when most other distinguishing features of these offices have been abstracted away from.

A structure S is said to be *rigid* when the only isomorphism of S onto itself is that given by the identity mapping. When the structure of R is rigid, its various offices can be regarded as objects in their own right. Let $\tau(x,R)$ be the abstract office occupied by x in R. Such abstract offices can then be individuated in the following way:

2. $\tau(x,R) = \tau(x',R') \leftrightarrow \exists f[f\colon R \cong R' \wedge f(x) = x']$.

For any realization R of a rigid abstract structure S, there is a unique isomorphism between the occupants of the offices of R and the abstract offices of the structure S.[31] This allows me to define a natural *predicational interpretation* of an abstract structure S. I can define S to hold of the abstract offices $\tau(x_1, R_1)$, ..., $\tau(x_n, R_n)$ iff there is a realization R of the abstract structure S which holds of u_1, ..., u_n, where for each i the office occupied by u_i in R corresponds to that occupied by x_i in R_i. This definition is permissible because it does not depend on the choice of realizations R and R_1, ..., R_n of the abstract structure S.

The dependence claim appears to be true of structures obtained by this form of Dedekind abstraction. In particular, the abstract offices appear to depend on the structure to which they belong: each such office has its identity solely in virtue of belonging to this particular structure. (This argument will be reinforced in the next section when the notion of dependence is analysed.)

As far as I can see, the only interesting anti-structuralist response to this argument would be to deny that there really are mathematical objects of the sort described in this section. It would take me too far afield to defend my argument against this challenge. I merely note that the structuralists have developed a variety of arguments to show that such mathematical objects should be accepted. Particularly powerful are the arguments to the effect that ordinary mathematical practice is committed to such objects.[32]

To sum up, the picture that emerges from this section and the previous one is that the dependence claim is true of some mathematical objects and false of others. This raises the question where to draw the line. Calling a

[31] What about the positions of non-rigid abstract structures? Are they too objects? If so, how are they individuated? For the purposes of establishing my compromise view, I need not take a stand on these difficult questions. For discussion, see J. Keränen, 'The Identity Problem for Realist Structuralism', *Philosophia Mathematica*, 9 (2001), pp. 308–30; Shapiro, 'The Governance of Identity', in F. MacBride (ed.), *Identity and Modality* (Oxford: Clarendon Press, 2006), pp. 164–73.

[32] See, e.g., Parsons, 'The Structuralist View of Mathematical Objects'; Shapiro, *Philosophy of Mathematics*, ch. 3; Resnik, *Mathematics as a Science of Patterns*, ch. 10.

structure *algebraic* if it arises from the process of Dedekind abstraction, that is, if it is just the abstract form of some system of objects and relations, I have argued that the structuralists are partially right because the dependence claim gives a correct description of algebraic structures. Although such structures do not exhaust mathematics, they include the vast majority of structures studied in contemporary mathematics. This gives structuralism a central place in the interpretation of mathematics. However, I have also argued that the structuralists are wrong about (iterative) sets and that the scope of the dependence claim will have to be restricted accordingly.[33]

VII. ANALYSING THE NOTION OF DEPENDENCE

The notion of dependence on which I have relied thus far is problematic. The most common analysis of the notion of dependence is probably the straightforward modal one, according to which x depends on y if and only if it is impossible for x to exist without y existing as well. However, since pure mathematical objects are generally assumed to exist necessarily, this modal analysis is useless for my purposes. For instance, on this analysis the existence of 2 no more depends on that of 5 than on that of the empty set. So any notion of dependence suited to figure in the debate about structuralism will have to be more fine-grained. But unfortunately neither party to the debate about the dependence claim has been particularly forthcoming about how to understand the crucial notion of dependence.

The only feature of the notion that emerges as reasonably clear is that the dependence in question is supposed to be a matter of how one object depends on others *for its identity* (see §§III–IV above). This kind of dependence has been analysed by metaphysicians. Two of the most sophisticated analyses are due to Kit Fine and E.J. Lowe.

To explain Fine's analysis of the relation of dependence I first need to review his analysis of the notion of an essential property. A property F is *essential* to an object x if and only if x could not have been the object it is without possessing the property F. For instance, it is essential to Socrates that he is human, and it is essential to the natural number 3 that it is abstract. The notion of an essential property cannot be reduced to modal notions, say by defining a property F to be essential to an object x if and only if it is necessary that Fx (if x exists at all).[34] This is clear from the relation

[33] I am also sympathetic with Parsons' claim that structuralism is false of '*quasi*-concrete' mathematical objects, such as geometrical figures and linguistic types.

[34] See Fine, 'Essence and Modality', in J. Tomberlin (ed.), *Philosophical Perspectives, 8: Language and Logic* (Atascadero: Ridgeview, 1994), pp. 1–16.

between Socrates and his singleton. This relation holds in all possible worlds in which Socrates exists. But although it is essential to the singleton that it contains Socrates as an element, it is not essential to Socrates that he is an element of this singleton. Equipped with this analysis of essential properties, Fine says that an object x *ontologically depends* on another object y if and only if y is a constituent of some essential property of x. For instance, the singleton of Socrates depends on Socrates because the property of having Socrates as an element is essential to it. But there is no dependence in the reverse direction because the property of being an element of a certain singleton is not essential to Socrates.

According to Lowe, x depends for its identity on y if and only if there is a function f such that it is essential to x that $x = f(y)$. This yields the same verdict as Fine's on the above example. Since it is essential to the singleton of Socrates that it is the value of the singleton function applied to Socrates as argument, this singleton depends on Socrates. But since it is not essential to Socrates that he is the value of the sole-element-of function applied to the singleton as argument, there is no dependence in the reverse direction.

In fact, these two analyses draw on a shared underlying idea, namely, that an object depends for its identity on another object if and only if any individuation of the former object must proceed via the latter. (By 'individuation' is meant here an explanation of the identity of an entity; see Lowe, 'Individuation'.) Applied to sets, this yields the familiar conclusion. In order to individuate a set we would have to specify its elements. It is therefore impossible to individuate the singleton of Socrates without proceeding via Socrates himself. But if a person allows of any informative individuation at all, this individuation certainly need not proceed via any set.

My present goal, as I have said, is to make it plausible that there are natural and important dependence relations on which the claims made in §§V–VI are sustained. In pursuit of this goal I propose to make use of the following two definitions. I shall say that x *strongly depends* on y if and only if any individuation of x must proceed via y. This is just the shared underlying idea mentioned above. I shall say that x *weakly depends* on y if and only if any individuation of x must make use of entities which also suffice to individuate y. For instance, a set x weakly depends on each of its subsets. For any individuation of x must make use of x's elements. But these elements suffice to individuate any subset of x as well. This weak dependence relation has (as far as I know) received little or no attention, despite being closely connected with Fine's and Lowe's shared underlying idea. Although both notions need further explication to be fully satisfactory, they suffice for present purposes.

It is now clear that the claims made in §V are sustained on both the strong and the weak notion of dependence. I have already pointed out that a set is strongly dependent on its elements but that the elements are not strongly dependent on the set. But in fact the elements of a set are not even weakly dependent on the set. For in order to individuate a set, we need to draw on additional entities not needed to individuate its elements, namely, the elements themselves. Sets thus provide a counter-example to (ODO), for either notion of dependence. Sets also provide a counter-example to (ODS) for either notion of dependence. No set is strongly dependent on the structure of the entire universe of sets. For every set can be individuated without proceeding via this structure. Nor is any set weakly dependent on this structure. For the entities needed to individuate a set fall (quite literally) infinitely far short of those needed to individuate this structure.

In §VI I claimed that the offices of an abstract structure *S* depend on one another and on the structure *S*. It is now evident that these claims are sustained on the weak notion of dependence. For example, an abstract office can be individuated via an ordered pair $\langle x, R \rangle$, where *R* is some particular system that realizes the abstract structure *S*, and where *x* is an object in the field of *R*. As part of the system *R* we are also given occupants of all the other *R*-offices. We thus have available the entities needed to individuate any of the other abstract offices. This means that each office weakly depends on all the others. It can also be shown that every office of an abstract structure weakly depends on the structure itself. For in order to individuate such an office we need a realization of the structure. But this is also all we need to individuate the relevant abstract structure itself. (For completeness, I remark that the converse also holds. For the realization needed to individuate the abstract structure contains all the entities needed to individuate its abstract offices as well.)

I conclude that my counter-examples to the two extreme views on the dependence relations of mathematical objects are borne out on a plausible analysis of the notion of dependence.[35]

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